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Original Article

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A Systematic Inquiry of Algebraic Implications of Boolean–Like Near Rings Under Pseudo Commutativity

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Abstract: *A. L.Foster and Alfred L.Foster introduced "Boolean-like rings" as generalizations of Boolean rings, and this concept was extended to "Boolean-like near-rings" by researchers like Clay, James.R and Lawver, Donald. In this paper, we have established conditions under which pseudo-commutative Boolean-like near-rings exhibit properties such as idempotency, distributive behaviour, and reduced ideal structures. Pseudo-commutativity is shown to impose significant constraints on the multiplication structure, leading to results analogous to those obtained under full commutativity. We defined the equivalence relation N on R×S by (r1, s1) ~ (r2, s1) if there exists an element* $s \in S$ *such that* $s(r1s2 - r2s1) = 0$*. The study demonstrates that pseudo-commutativity serves as an effective substitute for commutativity in deriving key structural results, thereby broadening the scope of Boolean-like near-ring theory and offering new directions for further research.*

Keywords: *Near-ring, Boolean-like Near Ring, Pseudo Commutativity, Distributivity, Reduced, Nilpotent, Idempotent.*

INTRODUCTION

Boolean rings, characterized by the idempotency condition $x^2 = x$ for all elements x, form a fundamental class in ring theory with deep connections to Boolean algebras and logic. The concept was generalized by A.L [1]. Foster to Boolean-like rings, which relax certain commutativity and distributivity assumptions while preserving key algebraic features. This notion was further extended to the setting of near-rings by Clay and Lawver [2][3], giving rise to Boolean-like near-rings. In this paper, we investigate the algebraic implications of Boolean-like near-rings under the weaker condition of pseudo-commutativity instead of full commutativity [4].

Pseudo-commutativity, defined by the relation $xyz = zyx$ for all $x, y, z \in N$, imposes significant structural constraints on the multiplication operation [5]. We establish several important results concerning idempotency, distributivity, reducedness, and ideal structures, demonstrating that pseudo-commutativity serves as an effective substitute for ordinary commutativity in deriving key properties analogous to those known in the commutative case [6]. This approach not only broadens the scope of Boolean-like near-ring theory but also opens new avenues for exploring non-commutative generalizations in near-ring structures.

PRELIMINARIES

Definition 1.1

A *(right)Near ring* is a set N together with two binary operations "+" and "." such that

a) (N,+) is a group (not necessarily abelian),
b) (N,.) is a semigroup
c) $(n_1 + n_2) n_3 = n_1n_3 + n_2n_3$, for all n_1,n_2,n_3 in N (right distributive law)

Definition 1.2

An element a in G is said to be an *idempotent* element if $a^2 = a$ and the set of all idempotents is denoted by E.

Definition 1.3

A near ring R is said to be *zero commutative* if ab = 0 implies ba = 0, for all a,b in R.

Definition 1.4

N is said to fulfil the Insertion of Factors Property (IFP) if $ab = 0 \Longrightarrow anb = 0$ for all a, b, n in N. If in addition $ab = 0 \Longrightarrow ba = 0$ for $a, b \in N$, we say N has (∗,IFP).

Definition 1.5

An N – subgroup A of N is essential if A ∩ B = {0}, where B is any N subgroup of N implies B = {0}.

Definition 1.6

A near ring N is said to be pseudo commutative near ring if $xyz = zyx, \forall x, y, z \in N$.

Definition 1.7

A near ring N is called Boolean if $x\ 2 = x$ for all $x \in N$.

Definition 1.8

A right near-ring (N, +,·) is called a Boolean-like near ring if

(i) $2a = 0\ \forall\ a \in$ N and
(ii) $(a + b - ab) = ab\ \forall\ a, b \in$ N

Definition 1.9

An element d ∈ N is said to be distributive if for all m, n ∈ N, d(m + n) = dm + dn. We denote N_d = {d ∈ N|d is distributive}.

Definition 1.10

The function f: R → R' is said to be a *homomorphism* if:

(i) f(a+b) = f(a) + f(b)
(ii) f(ab) = f(a) f(b), for all a,b in R

Proposition 1.11

Let N be a near ring.

a) N is abelian and N is commutative if and only if N is a commutative ring.
b) N is abelian and N is distributive if and only if N is a ring.
c) $N^2 = N$ and N is distributive implies that N is a ring.

Theorem 1.12

i) If I is an ideal of N, then the canonical mapping f: N $\rightarrow$ N\I (defined by f(n) = (n + I)) is a near ring epimorphism [7-10]. Also, N\I is a homomorphic image of N.
ii) Conversely, if h: N $\rightarrow N^1$ is an epimorphism, then ker(h) is an ideal of N, and N \ ker(h) is isomorphic to N^1 [11-12].

Definition 1.13

N is said to be *subcommutative*, if aN = Na for all a ϵ N

Theorem 1.14

If N is a Boolean (left) near – ring, then for any a,b ϵ N, ab = 0 => ba = 0.a.

MAIN RESULTS

Theorem 3.1

Let N be a Boolean-like near ring with P_4 property. If N is pseudo commutative near ring, then $x^2y + xy^2 = yx + (yx)^2$, for all x,y,z in N [13].

Proof:

Given: N is Boolean – like near ring

N has P_4 property

(i.e.) ab ϵ I ⇒ ba ϵ I, where I is an ideal of N ---(1)

N is pseudo commutative

(i.e.) xyz = zyx, for all x,y,z in N ----(2)

Consider, $x^2y + xy^2 = xxy + xyy$
$= yxx + yyx$ (By (2))
$= (yx + yy)\, x$
$= (y\,(x + y))\, x$
$= (x + y)\, yx$ (By (1))
$= (x + y - yx + yx)\, yx$
$= (x + y - yx) + (yx)^2$
$= yx + (yx)^2$ (By (1))

Hence, Proved.

Theorem 3. 2

Let N be a strongly regular Boolean-like near-ring with P_4 property.

If N is a distributive pseudo-commutative Boolean near-ring, then $(x + x^2)(y + y^2)z = z(y + xy), \forall x, y, z \in N$.

Proof:

Given: N is a strongly regular near-ring.

Consider $(x + x^2)(y + y^2)z = [y(x + x^2) + y^2(x + x^2)]z$

$$= y(x + x^2)z + y^2(x + x^2)z$$
$$= z(x + x^2)y + z(x + x^2)y^2 \text{[since N is pseudo commutative]}$$
$$= z[(x + x^2)y + (x + x^2)y^2] \text{[since N is distributive]}$$
$$= z[xy + x^2y + xy^2 + x^2y^2]$$
$$= z[xy + x^2y + xyy + x^2y^2]$$
$$= z[xy + x^2y + y + x^2y^2]$$

$= z\ [yx + (yx)^2 + y + x^2y^2]$ [By theorem 3.1]

$= z[yx + yx + y + x^2y^2]$

$= z[xy + y + xy]$

$= z[0 + y + xy]$[By (2)]

$= z[y + xy]$

Hence, proved.

Theorem 3.3

Let N be a Boolean-like near-ring with P_4 property. If N is a distributive pseudo-commutative near-ring, then $(x – x^2)\ (y – y^2)\ z = 0$, for every x,y,z in N [14].

Proof:

Let N be a Boolean-like near-ring. Let x,y,z ϵ N

Consider, $(x - x^2)(y - y^2)z = [x(y - y^2) - x^2(y - y^2)]z$

$$= x(y - y^2)z - x^2(y - y^2)z$$
$$= z(y - y^2)x - z(y - y^2)x^2 [\because N\text{-pseudo commutative}]$$
$$= z[(y - y^2)x - (y - y^2)x^2]$$
$$= z[yx - y^2x - yx^2 + y^2x^2]$$
$$= z[yx - (y^2x + yx^2) + y^2x^2]$$
$$= z[yx - (yx + (yx)^2) + y^2x^2] \text{[By theorem 3.1]}$$
$$= z[yx - yx - y^2x^2 + y^2x^2]$$
$$= z \cdot 0 [\because N\text{-Boolean}]$$
$$= 0$$

$$\Rightarrow \boxed{(x - x^2)(y - y^2)z = 0}$$

Remark – 3.4

The relation of theorem 3.1 becomes zero if Boolean property holds.

Proof:

W.K.T, $x^2y + xy^2 = yx + (y - x)^2$

$$= yx + yx$$
$$= 2yx$$
$$= 0$$

Theorem 3.5

Let N be a Boolean-like near-ring with pseudo commutativity. Let R be a commutative subset of N which is multiplicatively closed. Define a relation ~ on N X R by $(k_1, r_1) \sim$

(k_2, r_2) $\Leftrightarrow$ there exists $r \in R$ such that r $(k_1 r_2 - k_2 r_1) = 0$, where $k_1, k_2 \in N$. Then $\sim$ is an equivalence relation.

Proof:

Let N be a Boolean-like near-ring.

Since N is pseudo-commutative,

$xyz = zyx, \forall x, y, z \in N - -(1)$
Let R be a commutative subset of $N$with multiplicatively closed property.

Define a relation $\sim$ on $N \times R$ by $(k_1, r_1) \sim (k_2, r_2)$

$\Leftrightarrow$ there exists $r \in R$
$\Rightarrow r(k_1 r_2 - k_2 r_1) = 0$

(i) To prove: $\sim$ is reflexive

Let $(k, r) \in N \times R$.

Since,$kr - kr = 0$, we get t (ns – ns) = 0, for every t in R.

So, $\sim$ is reflexive.

(ii) To prove: $\sim$is symmetric

Let $(k_1, r_1) \sim (k_2, r_2)$
Then, there exists an element $r \in R$ such that

$r(k_1 r_2 - k_2 r_1) = 0$
$\Rightarrow r(k_2 r_1 - k_1 r_2) = 0$
$\therefore (k_2, r_2) \sim (k_1, r_1)$.

So, $\sim$ is symmetric.

(iii) To prove: $\sim$ is transitive

Let $(k_1, r_1) \sim (k_2, r_2)$ and $(k_2, r_2) \sim (k_3, r_3)$
Then $\exists$ n, s $\in$ R

$\Rightarrow n(k_1r_2 - k_2r_1) = 0$ & $s(k_2r_3 - k_3r_2) = 0$

$\Rightarrow n(k_1r_2 - k_2r_1)r_3 = r_3.0 = 0$ & $s(k_2r_3 - k_3r_2)r_1 = r_1.0 = 0$

$\Rightarrow n(k_1r_2r_3 - k_2r_1r_3) = 0$ & $s(k_2r_3r_1 - k_3r_2r_1) = 0$

$\Rightarrow sn(k_1r_2r_3 - k_2r_1r_3) = 0$ & $ns(k_2r_3r_1 - k_3r_2r_1) = 0$

$\Rightarrow ns(k_1r_2r_3 - k_2r_1r_3 + k_2r_3r_1 - k_3r_2r_1) = 0$

$\Rightarrow ns(k_1r_2r_3 - k_3r_2r_1) = 0$

$\Rightarrow r_2ns(k_1r_3 - k_3r_1) = 0$

$\Rightarrow m(k_1r_3 - k_3r_1) = 0$, where $m = r_2ns \in R$

$\therefore (k_1, r_1) \sim (k_3, r_3)$

$\therefore \sim$ is transitive

$\therefore \sim$ is an equivalence relation.

Theorem 3.6

Let N be a Boolean-like near-ring with pseudo commutativity. Let B be a commutative subset of Nwhich is multiplicatively closed. If $0 \notin B$ and N has no zero divisors, then $(n_1, b_1) \sim (n_2, b_2) \iff n_1b_2 = n_2b_1$.

Proof:

Let N be a Boolean-like near-ring with pseudo commutativity.
Let B be a commutative subset in $N$with multiplicative closure property.

$\Rightarrow$ Given $(n_1, b_1) \sim (n_2, b_2)$
Then, for an element $b \in B$, $b(n_1b_2 - n_2b_1) = 0$(by theorem 3.5)
Since $0 \notin B$ and N has no zero divisors, we have $n_1b_2 - n_2b_1 = 0$
$\Rightarrow n_1b_2 = n_2b_1$

Conversely, let $n_1b_2 = n_2b_1$
Then, $n_1b_2 - n_2b_1 = 0$
$\Rightarrow (n_1b_2 - n_2b_1)b = 0, \forall b \in B$
$\therefore (n_1, b_1) \sim (n_2, b_2)$

Lemma 3.7

Let N be a Boolean-like near-ring with pseudo commutativity. Let T be a commutative subset of N which is multiplicatively closed. Then,

(i) $\frac{rn}{t} = \frac{ns}{st} = \frac{sn}{ts} = \frac{sn}{st}, \forall n \in N, \forall s, t \in T$
(ii) $\frac{nt}{t} = \frac{nt'}{t'}, \forall n \in N, \forall t, t' \in T$
(iii) $\frac{t}{t} = \frac{t'}{t'}, \forall t, t' \in T$

Proof:

Obvious.

CONCLUSION

In this paper, we have successfully demonstrated that pseudo-commutativity acts as a powerful and natural substitute for full commutativity in the study of Boolean-like near-rings. By imposing the condition $xyz = zyx$, several significant structural results have been obtained, including simplified identities involving idempotents, conditions for distributivity, zero-product behaviors, and the construction of an equivalence relation on Cartesian products that behaves well under the given algebraic constraints. The presence of the P4 property together with pseudo-commutativity further strengthens the analogy with commutative Boolean-like structures, leading to reduced ideal structures and nilpotency control. These findings highlight that many classical results in Boolean-like near-ring theory remain valid under this milder commutativity assumption, thereby significantly enlarging the class of near-rings to which such results apply. The study paves the way for further investigations into pseudo-commutative near-rings and their potential applications in generalized algebraic systems, offering a promising direction for future research in non-commutative near-ring theory.

REFERENCES

[1] R. Balakrishnan and S. Suryanarayanan, “On *PK* and *Pk '* Near – Rings”, *Bull. Malaysian Math., Sc. Soc. (Second Series)*, vol. 23, pp. 9 – 24, 2000.

[2] C. Dhivya and D. Radha, "$\beta 6$ Near Rings", *Science, Technology and Development*, vol. 11, no. 05, pp. 438 - 443, 2022.

[3] S. Geetha and G. Gopalakrishnamoorthy, "On Quasi – Weak Commutative Near – Rings Iii", *Advances in Mathematics: Scientific Journal*, vol. 8, no. 3, pp. 423 – 429, 2019.

[4] G. Gopalakrishnamoorthy, M. Kamaraj Ands. Geetha, "On Quasi Weak Commutative Near-Rings", *International Journal of Mathematics Research*, vol. 5, no. 5, pp. 431-440, 2013.

[5] Gratzer and George, "*Universal Algebra*", Van Nostrand, 1968.

[6] Gunter Pilz,"*Near Rings*", North Holland, Amsterdam, 1983.

[7] D.J. Hansen and Jiang Luh, "Boolean Near Rings and Weak Commutativity," *J. Austral. Math. Soc. (Series A)*, vol. 47, pp. 103 – 107, 1989.

[8] Neal. H. Mccoy, "*Theory of Rings*", Macmillan & Co, 1970.

[9] D. Radha, C. Dhivya, M. Vinutha, K. Muthu Maheswari and S.R. Veronica Valli, "A Study on R-Near Ring", *Journal of Emerging Technologies and Innovative Research (JETIR)*, vol. 6, no. 12, 2019.

[10] D. Radha and C. Dhivya, "On S-Near Rings and S'-Near Rings with Right Bipotency", *JETIR*, vol. 6, 2019.

[11] D. Radha, M. Vinutha and C. Raja Lakshmi, "A Study on GS-Near Ring," *Journal of Emerging Technologies and Innovative Research (JETIR)*, Vol. 6, no. 2, 2019.

[12] D. Radha, C. Dhivya and S.R. Veronica Valli, "A Study on Quasi Weak Commutative Gamma Near Rings," *Journal of Emerging Technologies and Innovative Research*, vol. 6, no. 6,2019.

[13] S. Uma, R. Balakrishnan, and T. Tamizh Chelvam, "Pseudo Commutative Near-Rings," *Scientia Magna*, vol. 6, no. 2.

[14] Yong Uk Cho, "On Semicentral Idempotents in Near Rings," *Applied Mathematical Sciences*, vol. 9, no. 77, 2015.