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Research Article

Exploring the Galerkin Method for the Numerical Solution of Differential Equations using Bernoulli Wavelets

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ABSTRACT: Differential equations are important in many different aspects of modern life. Differential equations, for instance, can be used in physics to model the trajectory of a projectile or the motion of particles in a fluid. This paper proposes the Galerkin technique for using Bernoulli wavelets to numerically solve differential equations. This technique helps us find numerically such types of problems using weight functions known as Bernoulli wavelets, which are assumed basis elements. The exact solution and the current approaches (FDM & LWGM) are contrasted with the numerical solutions obtained using the proposed method. Here we will use some test problems to demonstrate how well the proposed technique works.

KEYWORDS: Galerkin Technique, Bernoulli Wavelets, Differential Equations, Function Approximation, Finite Difference method (FDM).

INTRODUCTION

Finding the precise solutions to differential equations using traditional methods like calculus and trigonometry is necessary to understand how they behave. One can learn how differential equations behave under various conditions by using these solutions. Analytical methods are those that are used to calculate the exact solution. However, the main draw of numerical techniques is their ability to provide solutions for a wide range of differential equations that are not yet ready for analytical solutions. In the literature, differential equations have recently been solved numerically using a few different numerical techniques [1-3].

In many fields, such as electrical engineering, seismic geology, quantum physics, and mathematics, wavelets have become autonomous ideas. An essential concept in approximation theory is the representation of a smooth function as a series expansion using orthogonal polynomials. In wavelet theory, wavelet function bases are being investigated as a potential substitute for piecewise polynomial trial functions for solving differential equations numerically. In applied mathematics, the Galerkin technique is highly regarded due to its usefulness and effectiveness [4]. With its many advantages over the conventional finite difference and finite element methods, the Galerkin method with wavelets finds widespread use in a variety of scientific and engineering domains. To a certain extent, the wavelet approach is a powerful alternative to the finite element method. Furthermore, the wavelet technique is an effective alternative to the numerical solution of differential equations [5–6]. This study presents the Wavelet-based Galerkin technique, which utilizes Bernoulli wavelets (BWGM) to solve differential equations numerically.

The present paper is organized as follows: The Bernoulli wavelets and function approximation are given in Sec. 2. Section 3 provides the Bernoulli wavelet based Galerkin technique (BWGM) for solving differential equations. Section 4 presents the numerical illustration. Finally, the conclusion of the proposed work is discussed in Sec. 5.

BERNOULLI WAVELETS AND FUNCTION APPROXIMATION

Bernoulli wavelets

Bernoulli wavelets $\psi_{i,j}(x)=\psi(k,\hat{i},j,x)$ have four arguments: $\hat{i} = i - 1, i = 1,2,...,2^{k-1}$, k is assumed to be any positive integer, j is the order for Bernoulli polynomials, and x is the normalized time [7 – 8]. They are defined on the interval $[0,1)$ as follows:

$$\psi_{i,j}(x) = \begin{cases} 2^{\frac{k-1}{2}}\tilde{B}_j(2^{k-1}x - i + 1), & \frac{i-1}{2^{k-1}} \le x < \frac{i}{2^{k-1}}, \\ 0, & \text{otherwise,} \end{cases} \tag{2.1}$$

with

$$\tilde{B}_j(x) = \begin{cases} 1, & j = 0 \\ \dfrac{1}{\sqrt{\left(\dfrac{(-1)^{j-1}(j!)^2}{(2j)!}\right)\alpha_{2j}}}, & j > 0 \end{cases} \tag{2.2}$$

where $j = 0,1,2,...,M-1$ and $i = 1,2,...,2^{k-1}$. The coefficient $\dfrac{1}{\sqrt{\left(\dfrac{(-1)^{j-1}(j!)^2}{(2j)!}\right)\alpha_{2j}}}$ is for normality, the dilation parameter is $a=2^{-(k-1)}$ and the translation parameter is $b=(i-1)2^{-(k-1)}$. Here, $B_j(x) = \sum_{n=0}^{j}\binom{j}{n}\alpha_{j-1}x^n$ are the Bernoulli polynomials of order m and α_i, $i = 0,1,2,.....,m$ are Bernoulli numbers. These numbers are a sequence of signed rational numbers that arise in the series expansion of trigonometric functions and can be defined by the identity $\dfrac{x}{e^x - 1} = \sum_{n=0}^{j}\alpha_n\dfrac{x^n}{n!}$.

The first few Bernoulli numbers are $\alpha_0 = 1$, $\alpha_1 = -\frac{1}{2}$, $\alpha_2 = \frac{1}{6}$, ... and Bernoulli polynomials are:

$B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}$, ...

The Bernoulli wavelet bases for $k = 1$ & $M = 3$ are:

$\psi_{1,0}(x) = 1$, $\psi_{1,1}(x) = \sqrt{3}\,(2x-1)$, $\psi_{1,2}(x) = \sqrt{5}\left(6x^2 - 6x + 1\right)$ and so on.

Function approximation:

Suppose $y(x)$ is expanded using Bernoulli wavelets in $L^2[0\ ,\ 1)$ as:

$$y(x) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} c_{i,j} \psi_{i,j}(x) \tag{2.3}$$

Approximate $y(x)$ by truncating the above infinite series i.e.

$$y(x) = \sum_{i=1}^{2^{k-1}} \sum_{j=0}^{M-1} c_{i,j} \psi_{i,j}(x) \tag{2.4}$$

METHOD OF SOLUTION

The general form of differential equation (D.E.) is,

$$y'' + \alpha y' + \beta y = f(x) \tag{3.1}$$

Boundary conditions $y(0) = a,\ y(1) = b$ (3.2)

Here α & β are constants and $f(x)$ be a continuous function.

Write the Eq. (3.1) as $R(x) = y'' + \alpha y' + \beta y - f(x)$ (3.3)

Here $R(x)$ is the residual of Eq. (3.1). If $R(x) = 0$ for the exact solution and $y(x)$ will satisfy the boundary conditions.

Take the trail series solution of Eq. (3.1), $y(x)$ defined over [0, 1) which can be expanded using modified Bernoulli wavelets and satisfying the given boundary conditions which involve unknown coefficients as follows,

$$y(x) = \sum_{i=1}^{2^{k-1}} \sum_{j=0}^{M-1} c_{i,j} \psi_{i,j}(x) \tag{3.4}$$

The unknown coefficients $c_{n,m}$'s are to be determined.

By choosing higher degree Bernoulli wavelet polynomials, accuracy in the solution is increased. Differentiate Eq. (3.4) twice with respect to x and substitute the values of y , y' , y'' in Eq. (4.3). We choose weight functions that are assumed base elements and integrate boundary values with the residual to zero for finding the unknown coefficients $c_{n,m}$'s [8].

i.e. $$\int_0^1 \psi_{1,j}(x)\ R(x)\ dx = 0,\ j = 0, 1, 2, ...$$

From the above integral, a system of linear algebraic equations is obtained and solving this system, is to get unknown coefficients. Substitute these unknown coefficients in Eq. (3.4), to obtain the numerical solution of Eq. (3.1). To know the accuracy of BWGM for the test problems, use the error measure i.e. maximum absolute error and it will be calculated by $E_{\max} = \max \left| y(x)_e - y(x)_a \right|$, $y(x)_e$ and $y(x)_a$ are exact and approximate solutions respectively.

NUMERICAL ILLUSTRATION

Problem 4.1 Consider the D.E. [3],

$$y'' - y' = -1,\ 0 \le x \le 1 \tag{4.1}$$

$$\text{BCs:} \quad y(0) = 0, \quad y(1) = 0 \tag{4.2}$$

As per the method explained in section 3 and the implementation for the solution of the Eq. (4.1) is as follows:

The residual of Eq. (4.1) can be written as:

$$R(x) = y'' - y' + 1 \tag{4.3}$$

By choosing the weight function $w(x) = x(1-x)$ for Bernoulli wavelet bases to satisfy the given boundary conditions i.e. Eq. (4.2), i.e. $\psi(x) = w(x) \times \psi(x)$

$$\psi_{1,0}(x) = \psi_{1,0}(x) \times x(1-x) = x(1-x)$$

$$\psi_{1,1}(x) = \psi_{1,1}(x) \times x(1-x) = \sqrt{3}\,(2x-1)x(1-x)$$

$$\psi_{1,2}(x) = \psi_{1,2}(x) \times x(1-x) = \sqrt{5}\left(6x^2 - 6x + 1\right)x(1-x)$$

Consider the trail solution of Eq. (4.1) for $k = 1$ and $M = 3$ is given by

$$y(x) = c_{1,0}\,\psi_{1,0}(x) + c_{1,1}\,\psi_{1,1}(x) + c_{1,2}\,\psi_{1,2}(x) \tag{4.4}$$

Eq. (4.4) becomes

$$y(x) = c_{1,0}\{x(1-x)\} + c_{1,1}\left\{\sqrt{3}\,(2x-1)x(1-x)\right\} + c_{1,2}\left\{\sqrt{5}\left(6x^2 - 6x + 1\right)x(1-x)\right\} \tag{4.5}$$

Differentiate Eq. (4.5) twice w.r.t. x and substitute the values of y', y'' in Eq. (4.3) and obtain the residual of Eq. (4.1). The "weight functions" are the same as the basis functions.

By the weighted Galerkin method, consider the following:

$$\int_0^1 \psi_{1,j}(x)\,R(x)\,dx = 0, \quad j = 0, 1, 2 \tag{4.6}$$

For $j = 0, 1, 2$ in Eq. (4.6),

$$\text{i.e.} \quad \left.\begin{aligned} \int_0^1 \psi_{1,0}(x)\,R(x)\,dx &= 0 \\ \int_0^1 \psi_{1,1}(x)\,R(x)\,dx &= 0 \\ \int_0^1 \psi_{1,2}(x)\,R(x)\,dx &= 0 \end{aligned}\right\} \tag{4.7}$$

From Eq. (4.7), the system of algebraic equations with unknown coefficients is found, and solving this system, the values for $c_{1,0} = 0.4934$, $c_{1,1} = 0.0470$ and $c_{1,2} = 0.0031$. Substitute these values in Eq. (4.5), and the numerical solution for Eq. (4.1) is obtained. A comparison of the numerical solution and the absolute errors is presented in Table 1 and the numerical solution with the exact solution of Eq. (4.1) is $y(x) = x - \left(\dfrac{e^x - 1}{e - 1}\right)$ shown in Figure 1.

Table 1: Numerical solution and absolute error with exact solution of the problem 4.1

x	Numerical solution			Exact solution	Absolute error		
	FDM	Ref [3]	BWGM		FDM	Ref [3]	BWGM
0.1	0.037255	0.038684	0.038832	0.038793	1.54e-03	1.09e-04	3.90e-05
0.2	0.068235	0.071099	0.071173	0.071149	2.91e-03	5.00e-05	2.40e-05
0.3	0.092313	0.096232	0.096397	0.09639	4.08e-03	1.58e-04	7.00e-06
0.4	0.108799	0.113656	0.113777	0.113769	4.97e-03	1.13e-04	8.00e-06
0.5	0.116933	0.12242	0.122484	0.122459	5.53e-03	3.90e-05	2.50e-05
0.6	0.115881	0.121367	0.121592	0.121546	5.66e-03	1.79e-04	4.60e-05
0.7	0.104724	0.109825	0.110074	0.11002	5.30e-03	1.95e-04	5.40e-05
0.8	0.082451	0.086853	0.086803	0.086764	4.31e-03	8.90e-05	3.90e-05
0.9	0.04795	0.050414	0.050554	0.050545	2.59e-03	1.31e-04	9.00e-06

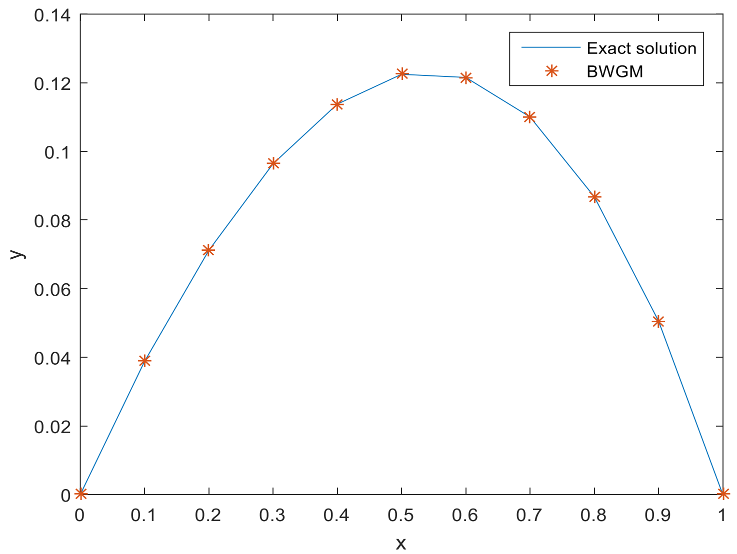

Figure 1: Numerical solution (BWGM) with exact solution of the problem 4.1.

Problem 4.2 Next, another D.E. [9],

$$y'' + \frac{16}{9}\pi^2 y = \frac{7}{9}\pi^2 \sin(\pi x), \quad 0 \le x \le 1 \tag{4.8}$$

$$\text{BCs:} \quad y(0) = 0, \quad y(1) = 0 \tag{4.9}$$

As per section 3, the values of $c_{1,0} = 3.7028$, $c_{1,1} = 0.0$ and $c_{1,2} = -0.2629$ are determined. The numerical solution was then derived by substituting the values of $c_{1,0}$, $c_{1,1}$ and $c_{1,2}$ in Eq. (4.5). Table 2 compares the numerical solution to the absolute errors, whereas Figure 2 comparison of the numerical solution to the exact solution of Eq. (4.8) $y(t) = \sin(\pi t)$.

Table 2: Numerical solutions with the exact solution and the absolute errors for test problem 4.2

x	Numerical Solution		Exact solution	Absolute error	
	Ref [9]	BWGM		Ref [9]	BWGM
0.1	0.3087468	0.3089145	0.309016	2.69e-04	1.02e-04
0.2	0.5925196	0.5886857	0.588772	3.75e-03	8.63e-05
0.3	0.8151813	0.8096853	0.809016	6.16e-03	6.93e-04
0.4	0.9540854	0.9507503	0.951056	3.03e-03	3.06e-04
0.5	0.9982500	0.9991828	1.000000	1.75e-03	8.17e-04
0.6	0.9465312	0.9507503	0.951056	4.52e-04	3.06e-04
0.7	0.7952968	0.8096853	0.809016	1.37e-02	6.69e-04
0.8	0.5811001	0.5886857	0.587785	6.68e-03	9.01e-04
0.9	0.3093530	0.3089145	0.309016	3.37e-04	1.02e-04

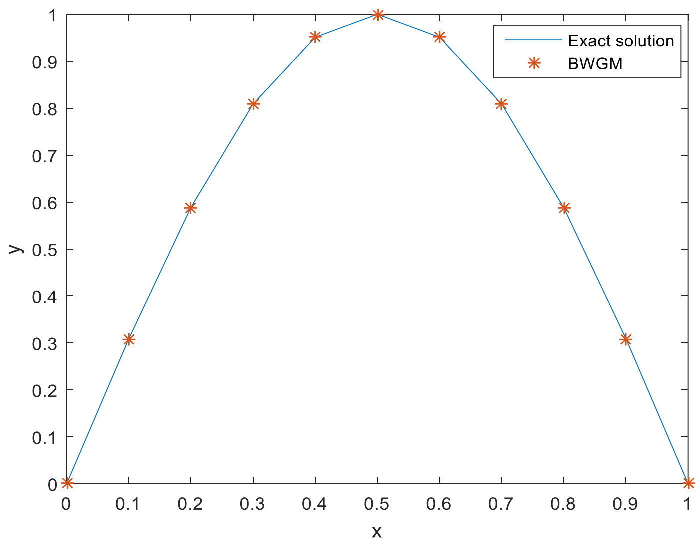

Figure 2: Numerical solution (BWGM) with exact solution of the problem 4.2.

Problem 4.3 Finally, another D.E. [10],

$$y'' - 4y = 4\cosh(1), \quad 0 \le x \le 1 \tag{4.10}$$

$$\text{BCs:} \quad y(0) = 0, \quad y(1) = 0 \tag{4.11}$$

As per section 3, the values i.e. $c_{1,0} = -2.2306$, $c_{1,1} = 0.0$ and $c_{1,2} = -0.0525$. To obtain Eq. (4.10), substitute these values of $c_{1,0}$, $c_{1,1}$ and $c_{1,2}$ in Eq. (4.5). A comparison of the numerical solution to the absolute errors in Table 3 as well as Figure 3 shows the numerical solution to the exact solution of Eq. (4.10) $y(x) = \cosh(2x - 1) - \cosh(1)$.

Table 3: Numerical solutions and absolute error with the exact solution for test problem 4.3

x	Numerical Solution		Exact solution	Absolute error	
	FDM	BWGM		FDM	BWGM
0.1	-0.20513451	-0.2056141	-0.2056457	5.11e-04	3.16e-05
0.2	-0.35675116	-0.3576473	-0.3576124	8.61e-04	3.49e-05
0.3	-0.46091565	-0.4620163	-0.4620083	1.09e-03	8.00e-06
0.4	-0.52179149	-0.5229472	-0.5230139	1.22e-03	6.67e-05
0.5	-0.54181677	-0.5429758	-0.5430806	1.26e-03	1.05e-04
0.6	-0.52179149	-0.5229472	-0.5230139	1.22e-03	6.67e-05
0.7	-0.46091465	-0.4620163	-0.4620083	1.09e-03	8.00e-06
0.8	-0.356751167	-0.3576473	-0.3576124	8.61e-04	3.49e-05
0.9	-0.20513451	-0.2056141	-0.2056457	5.11e-04	3.16e-05

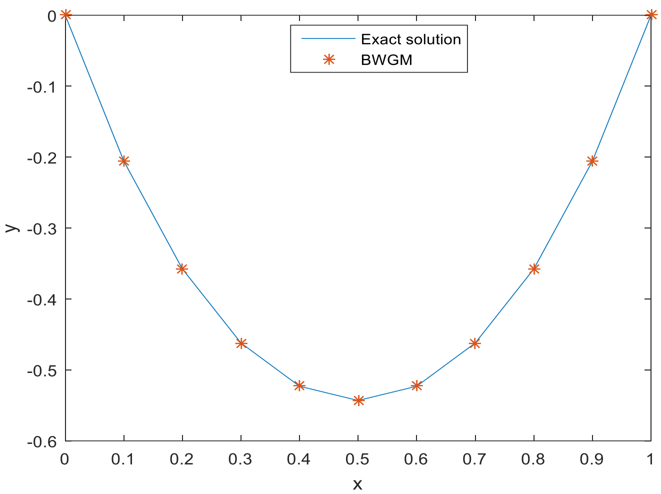

Figure 3: Numerical solution (BWGM) with exact solution of the problem 4.3

CONCLUSION

The Galerkin approach for using Bernoulli wavelets to numerically solve differential equations is presented in this study. The above tables and figures reveal that the numerical solutions derived from the proposed approach outperform those generated by the existing methods (FDM, Ref [3] & Ref [9]) and show a closer resemblance to the exact solution. Additionally, compared to the existing method (FDM, Ref [3] & Ref [9]), the absolute error associated with our methodology is substantially smaller. For differential equations, the Galerkin approach using Bernoulli wavelets is hence highly successful.

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