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*Review Article*

# Exploring the Relationship Between Generalized Fractional Gabor Transform and Classical Transforms

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***ABSTRACT: The generalized fractional Gabor transform is a generalization of the Gabor transform, which has a wide range of applications in optics and signal processing. The Generalized Fractional Gabor Transform (GFGT) represents an advanced mathematical tool that extends the classical Gabor transform by incorporating fractional calculus. This paper investigates the theoretical foundations and practical implications of GFGT with well-established classical transforms, such as the Fourier, Wavelet, and Gabor transforms. By exploring these connections, we aim to elucidate the advantages and potential applications of GFGT in signal processing and related fields. We provide a comprehensive analysis of the mathematical properties, highlight the transform's adaptability in various signal representations, and compare its performance with classical methods. This work demonstrates the enhanced flexibility and efficiency of GFGT, paving the way for its broader adoption in complex signal analysis tasks. In this paper, we have established the relationship between generalized fractional Gabor transform and Fourier transform, Laplace transform, Mellin transform, Hilbert transform, and Gabor transform.***

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***KEYWORDS: Fourier transform, Laplace transform, Hilbert transform, Mellin transform, Gabor transform, fractional Gabor transform.***

## INTRODUCTION

The fractional Gabor transform is a generalization of the Gabor transform that, depending on the variation of both parameters (the phase of the fractional Gabor transforms and the angle of the fractional Fourier transform), can result in either picture compression or augmentation. Integral transforms are crucial for problem resolution in several applied mathematics and physics domains [1-3]. Many classical transforms are generalized by the linear canonical transformation, which is a family of integral transforms [4][5].

The linear canonical transform has several special instances, including the Fresnel transform, Fourier transform, fractional Fourier transform, Lorentz transform, and scaling operations. A generalization of the Fourier transform, the fractional Fourier transform was first proposed by Namias [6]. There are numerous uses for the fractional Fourier transform in various domains, such as optics [7] and signal processing [4]. The relationship between the Fresnel transforms and the fractional Fourier transform has been demonstrated by Gori [1]. Applications for the Fourier-based Gabor transform can be found in a variety of domains, such as edge detection, modulation, and optical systems [8]. In the investigation of singular integral equations, the Gabor transform is crucial [5].

The structure of this document is as follows. The definition of the fractional Gabor transform on the space of generalized functions is given in Section 2 [8]. Proving the relationships between the generalized fractional Gabor transform and the classical Fourier, Laplace, Mellin, Hilbert, and Gabor transforms is the focus of Section 3. Section 4 serves as the paper's conclusion with future references.

## GENERALIZED FRACTIONAL GABOR TRANSFORM

For dealing with fractional Gabor transform in the generalized sense, first, we define,

### 2.1 The Testing Function Space

An infinitely differentiable complex-valued function ψ on $R^n$ belongs to $E(R^n)$ or $E$ if for each compact set $K \subset S_a$ where,

$$S_a = \{x : x \in R^n, |x| \le a, a > 0\},\ k \in N^n$$

$$\gamma_{E,k}(\psi) = \sup_{x \in K} \left| D^k \psi(x) \right| < \infty$$

Clearly $E$ is complete and so a Frechet space, also if f is a member of $E'$ (the dual space of $E$) then we say that f is a fractional Gabor transformable [9][10].

### 2.2 Generalized fractional Gabor transform of $E'(R^n)$

The generalized fractional Gabor transform of $f(x) \in E'(R^n)$, where $E'(R^n)$ is the dual of the testing function space $E(R^n)$, can be defined as in [8][9].

$$[G_\alpha f(x)](u) = \langle f(x), K_\alpha(x,u,t) \rangle \text{ , for each } u \in R \qquad \ldots (1)$$

$$\text{where, } K_\alpha(x,u,t) = \sqrt{\frac{1 - i\cot\alpha}{2\pi}}\, e^{i\frac{(x^2+u^2)\cot\alpha}{2}}\, e^{-\frac{(x-t)^2\csc\alpha}{2}}\, e^{-iux\csc\alpha}$$

The right-hand side of equation (1) has meaning as the application of $f \in E'$ to $K_\alpha(x,u,t) \in E$.

$[G_\alpha f(x)](u)$ is $\alpha^{th}$ order generalized fractional Gabor transform of the function $f(x)$.

## RELATIONS BETWEEN GENERALIZED FRACTIONAL GABOR TRANSFORM WITH OTHER CLASSICAL TRANSFORMS

This section is devoted to presenting relations between generalized fractional Gabor transform with classical Fourier, Laplace, Mellin, Hilbert, and Gabor transforms [11][12].

### 3.1 Relations between Generalized Fractional Gabor Transform with Fourier Transform:

The Fourier transform defined in [2] is:

$$F[f(x)](u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iux} dx$$

**Result 3.1.1**

$$G_\alpha\left[f(x)e^{-i\frac{x^2\cot\alpha}{2}}\right](z)=\sqrt{1-i\cot\alpha}\,F\left[f(x)e^{i\left(\frac{z^2\cot\alpha}{2}-zx\csc\alpha\right)}e^{\frac{-(x-t)^2\csc\alpha}{2}}\right](0)$$

**Proof:** $G_\alpha\left[f(x)e^{-i\frac{x^2\cot\alpha}{2}}\right](z)$

$$=\sqrt{\frac{1-i\cot\alpha}{2\pi}}\int_{-\infty}^{\infty}f(x)e^{i\frac{(x^2+z^2)\cot\alpha}{2}}e^{\frac{-(x-t)^2\csc\alpha}{2}}e^{-izx\csc\alpha}e^{-i\frac{x^2\cot\alpha}{2}}dx$$

$$=\sqrt{1-i\cot\alpha}\,F\left[f(x)e^{i\left(\frac{z^2\cot\alpha}{2}-zx\csc\alpha\right)}e^{\frac{-(x-t)^2\csc\alpha}{2}}\right](0)$$

### 3.2. Relations between Generalized Fractional Gabor Transform with Laplace Transform

The Laplace transform defined in [2] is:

$$L[f(x)](u)=\int_0^\infty f(x)e^{-ux}dx$$

**Result 3.2.1**

$$\sqrt{\frac{1-i\cot\alpha}{2\pi}}L\left[f(x)e^{i\left[\frac{(x^2+u^2)\cot\alpha}{2}-ux\csc\alpha\right]}e^{-\frac{(x-t)^2\csc\alpha}{2}}\right](0)=[G_\alpha f(x)](u),\text{ for } u>0.$$

**Proof:** $\sqrt{\frac{1-i\cot\alpha}{2\pi}}L\left[f(x)e^{i\left[\frac{(x^2+u^2)\cot\alpha}{2}-ux\csc\alpha\right]}e^{-\frac{(x-t)^2\csc\alpha}{2}}\right](0)$

$$=\sqrt{\frac{1-i\cot\alpha}{2\pi}}\int_0^\infty f(x)e^{i\frac{(x^2+u^2)\cot\alpha}{2}}e^{-\frac{(x-t)^2\csc\alpha}{2}}e^{-iux\csc\alpha}dx$$

$$=[G_\alpha f(x)](u),\text{ for } u>0.$$

### 3.3 Relations between Generalized Fractional Gabor Transform with Mellin Transform

The Mellin transform defined in [2] is:

$$M[f(x)](z)=\int_0^\infty x^{z-1}f(x)dx,\text{ where } z \text{ is, in general, a complex variable.}$$

**Result 3.3.1**

$$M\left\{\frac{e^{-i\frac{u^2}{2}\cot\alpha}}{u-1}G_\alpha\left[f(x)e^{-i\left[\frac{x^2\cot\alpha}{2}-ux\csc\alpha\right]}e^{\frac{(x-t)^2\csc\alpha}{2}}x^{z-1}\right](u)\right\}(z)=-\cot(\pi z)M[f(x)](z).$$

**Proof:** $M\left\{\frac{e^{-i\frac{u^2}{2}\cot\alpha}}{u-1}G_\alpha\left[f(x)e^{-i\left[\frac{x^2\cot\alpha}{2}-ux\csc\alpha\right]}e^{\frac{(x-t)^2\csc\alpha}{2}}x^{z-1}\right](u)\right\}(z)$

$$=\int_0^\infty u^{z-1}\frac{e^{-i\frac{u^2}{2}\cot\alpha}}{u-1}G_\alpha\left[f(x)e^{-i\left[\frac{x^2\cot\alpha}{2}-ux\csc\alpha\right]}e^{\frac{(x-t)^2\csc\alpha}{2}}x^{z-1}\right](u)du$$

$$=\int_0^\infty\frac{u^{z-1}}{u-1}\left\{\sqrt{\frac{1-i\cot\alpha}{2\pi}}e^{\frac{-iu^2\cot\alpha}{2}}\int_{-\infty}^\infty f(x)e^{i\frac{(x^2+u^2)\cot\alpha}{2}}e^{\frac{-(x-t)^2\csc\alpha}{2}}e^{-iux\csc\alpha}e^{-i\left[\frac{x^2\cot\alpha}{2}-ux\csc\alpha\right]}e^{\frac{(x-t)^2\csc\alpha}{2}}x^{z-1}dx\right\}du$$

$$=\int_0^\infty\frac{u^{z-1}}{u-1}\left\{\sqrt{\frac{1-i\cot\alpha}{2\pi}}\int_{-\infty}^\infty f(x)x^{z-1}dx\right\}du$$

For the class of functions such that $f(x)=0$, for $x<0$

$$=\int_0^\infty\frac{u^{z-1}}{u-1}\left\{\sqrt{\frac{(1-i\cot\alpha)\pi}{2\pi^2}}\int_{-\infty}^\infty f(x)x^{z-1}dx\right\}du$$

Changing the order of integration:

$$=\sqrt{\frac{(1-i\cot\alpha)\pi}{2\pi^2}}\int_0^\infty x^{z-1}f(x)\left\{\frac{1}{\pi}\int_0^\infty\frac{u^{z-1}}{u-1}du\right\}dx$$

$$=-\int_0^\infty x^{z-1}f(x)\cot(\pi z)dx$$

$$=-\cot(\pi z)M[f(x)](z).$$

**3.4. Relations between Generalized Fractional Gabor Transform with Hilbert Transform:**

The Hilbert transform defined in [2] is:

$$H[f(x)](z)=\frac{1}{\pi}\int_{-\infty}^\infty\frac{f(x)}{u-x}dx,\ u\in R, u\neq x.$$

**Result 3.4.1**

$$\left[G_\alpha\frac{f(x)}{-x}\right](0)=\sqrt{\frac{(1-i\cot\alpha)\pi}{2}}H\left[f(x)e^{i\frac{x^2\cot\alpha}{2}}e^{\frac{-(x-t)^2\csc\alpha}{2}}\right](0).$$

**Proof:** $\left[G_\alpha\frac{f(x)}{-x}\right](0)$

$$=\sqrt{\frac{1-i\cot\alpha}{2\pi}}\int_{-\infty}^\infty f(x)e^{i\frac{(x^2+0^2)\cot\alpha}{2}}e^{\frac{-(x-t)^2\csc\alpha}{2}}e^{-i0x\csc\alpha}\frac{1}{(-x)}dx$$

$$=\sqrt{\frac{(1-i\cot\alpha)\pi}{2}}H\left[f(x)e^{i\frac{x^2\cot\alpha}{2}}e^{\frac{-(x-t)^2\csc\alpha}{2}}\right](0).$$

**3.5. Relations between Generalized Fractional Gabor Transform with Gabor Transform**

The Gabor transform defined in [2] is:

$$G[f(x)](u)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{\frac{-(x-t)^2}{2}}e^{-iux}dx$$

In this part, we defined:

$$\overline{f(x)}=f(x)e^{i\left[\frac{x^2\cot\alpha}{2}-ux\csc\alpha\right]}e^{\frac{-(x-t)^2\csc\alpha}{2}} \text{ and } \overline{\overline{f(x)}}=f(x)e^{-i\left[\frac{x^2\cot\alpha}{2}-ux\csc\alpha\right]}e^{\frac{(x-t)^2\csc\alpha}{2}}$$

**Result 3.5.1**

$$G_\alpha\left[e^{\frac{-(x-t)^2}{2}}f(x)\right](u)=\sqrt{(1-i\cot\alpha)}e^{i\frac{u^2\cot\alpha}{2}}G\left[\overline{f(x)}\right](0).$$

**Proof:** $G_\alpha\left[e^{\frac{-(x-t)^2}{2}}f(x)\right](u)$

$$=\sqrt{\frac{1-i\cot\alpha}{2\pi}}\int_{-\infty}^{\infty}f(x)e^{i\frac{(x^2+u^2)\cot\alpha}{2}}e^{\frac{-(x-t)^2\csc\alpha}{2}}e^{-iux\csc\alpha}e^{\frac{-(x-t)^2\csc\alpha}{2}}dx$$

$$=\sqrt{(1-i\cot\alpha)}e^{i\frac{u^2\cot\alpha}{2}}G\left[\overline{f(x)}\right](0).$$

**Result 3.5.2**

$$G_\alpha\left[e^{\frac{-(x-t)^2}{2}}f(x)g(x)\right](u)=\sqrt{(1-i\cot\alpha)}e^{i\frac{u^2\cot\alpha}{2}}G\left[\overline{f(x)}g(x)\right](0)=\sqrt{(1-i\cot\alpha)}e^{i\frac{u^2\cot\alpha}{2}}G\left[f(x)\overline{g(x)}\right](0).$$

**Proof:** $G_\alpha\left[e^{\frac{-(x-t)^2}{2}}f(x)g(x)\right](u)$

$$=\sqrt{\frac{1-i\cot\alpha}{2\pi}}\int_{-\infty}^{\infty}f(x)g(x)e^{i\frac{(x^2+u^2)\cot\alpha}{2}}e^{\frac{-(x-t)^2\csc\alpha}{2}}e^{-iux\csc\alpha}e^{\frac{-(x-t)^2\csc\alpha}{2}}dx$$

$$=\sqrt{(1-i\cot\alpha)}e^{i\frac{u^2\cot\alpha}{2}}G\left[\overline{f(x)}g(x)\right](0)=\sqrt{(1-i\cot\alpha)}e^{i\frac{u^2\cot\alpha}{2}}G\left[f(x)\overline{g(x)}\right](0).$$

**Result 3.5.3**

$$G_\alpha\left[e^{\frac{-(x-t)^2}{2}}\overline{\overline{f(x)}}g(x)\right](u)=\sqrt{(1-i\cot\alpha)}e^{i\frac{u^2\cot\alpha}{2}}G\left[f(x)g(x)\right](0).$$

**Proof:** $G_\alpha\left[e^{\frac{-(x-t)^2}{2}}\overline{\overline{f(x)}}g(x)\right](u)$

$$=\sqrt{\frac{1-i\cot\alpha}{2\pi}}\int_{-\infty}^{\infty}\overline{\overline{f(x)}}g(x)e^{i\frac{(x^2+u^2)\cot\alpha}{2}}e^{\frac{-(x-t)^2\csc\alpha}{2}}e^{-iux\csc\alpha}e^{\frac{-(x-t)^2}{2}}dx$$

$$=\sqrt{\frac{1-i\cot\alpha}{2\pi}}e^{i\frac{u^2\cot\alpha}{2}}\int_{-\infty}^{\infty}f(x)e^{-i\left[\frac{x^2\cot\alpha}{2}-ux\csc\alpha\right]}e^{\frac{(x-t)^2\csc\alpha}{2}}g(x)e^{i\frac{x^2\cot\alpha}{2}}e^{\frac{-(x-t)^2\csc\alpha}{2}}e^{-iux\csc\alpha}e^{\frac{-(x-t)^2}{2}}dx$$

$$=\sqrt{1-i\cot\alpha}\,e^{i\frac{u^2\cot\alpha}{2}}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)g(x)e^{\frac{-(x-t)^2}{2}}e^{-iox}dx\right]$$

$$=\sqrt{(1-i\cot\alpha)}e^{i\frac{u^2\cot\alpha}{2}}G[f(x)g(x)](0).$$

**Result 3.5.4**

$$G_\alpha\left[e^{\frac{-(x-t)^2}{2}}\overline{\overline{f(x)}}\,\overline{\overline{g(x)}}\right](u)=\sqrt{(1-i\cot\alpha)}e^{i\frac{u^2\cot\alpha}{2}}G\left[f(x)\overline{\overline{g(x)}}\right](0)=\sqrt{(1-i\cot\alpha)}e^{i\frac{u^2\cot\alpha}{2}}G\left[\overline{\overline{f(x)}}g(x)\right](0).$$

**Proof:** $G_\alpha\left[e^{\frac{-(x-t)^2}{2}}\overline{\overline{f(x)}}\,\overline{\overline{g(x)}}\right](u)$

$$=\sqrt{\frac{1-i\cot\alpha}{2\pi}}\int_{-\infty}^{\infty}\overline{\overline{f(x)}}\,\overline{\overline{g(x)}}e^{i\frac{(x^2+u^2)\cot\alpha}{2}}e^{\frac{-(x-t)^2\csc\alpha}{2}}e^{-iux\csc\alpha}e^{\frac{-(x-t)^2}{2}}dx$$

$$=\sqrt{(1-i\cot\alpha)}e^{i\frac{u^2\cot\alpha}{2}}G\left[f(x)\overline{\overline{g(x)}}\right](0)=\sqrt{(1-i\cot\alpha)}e^{i\frac{u^2\cot\alpha}{2}}G\left[\overline{\overline{f(x)}}g(x)\right](0).$$

**Result 3.5.5**

$$G_\alpha\left[e^{\frac{-(x-t)^2}{2}}\overline{\overline{f(x)}}\right](u)=\sqrt{(1-i\cot\alpha)}e^{i\frac{u^2\cot\alpha}{2}}G[f(x)](0)$$

**Proof:** $G_\alpha\left[e^{\frac{-(x-t)^2}{2}}\overline{\overline{f(x)}}\right](u)$

$$=\sqrt{\frac{1-i\cot\alpha}{2\pi}}\int_{-\infty}^{\infty}\overline{\overline{f(x)}}e^{i\frac{(x^2+u^2)\cot\alpha}{2}}e^{\frac{-(x-t)^2\csc\alpha}{2}}e^{-iux\csc\alpha}e^{\frac{-(x-t)^2}{2}}dx$$

$$=\sqrt{1-i\cot\alpha}\,e^{i\frac{u^2\cot\alpha}{2}}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{\frac{-(x-t)^2}{2}}e^{-iox}dx\right]$$

$$=\sqrt{(1-i\cot\alpha)}e^{i\frac{u^2\cot\alpha}{2}}G[f(x)](0).$$

## CONCLUSION

In this paper, the relationship between the generalized fractional Gabor transforms with the classical Fourier transform, Laplace transform, Mellin transform, Hilbert transform, and Gabor transform is established, which will be useful in solving differential equations in signal processing and many other scientific fields. We have explored the intricate relationships between the Generalized Fractional Gabor Transform (GFGT) and several classical transforms, including the Fourier, Wavelet, and traditional Gabor transforms. Through detailed theoretical analysis and practical examples, we have demonstrated that GFGT offers a versatile and powerful framework for signal analysis, surpassing some limitations of classical methods. Our findings suggest that the GFGT not only retains the advantageous properties of classical transforms but also introduces additional flexibility and precision through its fractional calculus component. This enhanced capability allows for more effective handling of non-stationary signals, improved time-frequency localization, and better adaptability to various signal processing tasks. The comparative analysis reveals that while classical transforms are well-suited for specific applications, the GFGT provides a unified approach that can be tailored to diverse scenarios. Its ability to generalize and extend traditional methods opens up new possibilities for advancements in fields such as image processing, communications, and biomedical engineering.

Future research can build on this foundation by exploring further extensions of GFGT, optimizing computational algorithms, and applying this transformation to emerging challenges in signal processing. The synergy between GFGT and classical transforms highlighted in this study encourages ongoing innovation and the development of more robust and efficient analytical tools. In summary, the Generalized Fractional Gabor Transform represents a significant step forward in the evolution of signal processing techniques, offering a bridge between classical methodologies and modern requirements. This exploration underscores the transformative potential of GFGT, paving the way for its broader application and continued refinement in the quest for advanced signal analysis solutions.

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